Duration Models and Matching

Stephen Pettigrew

April 8, 2015

Logistics

- Pset 6 due tonight. New pset posted tonight. Not due for two weeks (April 22)
- Submit a title and abstract for your final paper
  - One per group on the message board

Logistics

- Party at Gary's house!
  - Saturday, May 2 at noon
  - Please RSVP at the link included in last night's email
  - Near the Green Line and the 86 bus that goes through Harvard Square

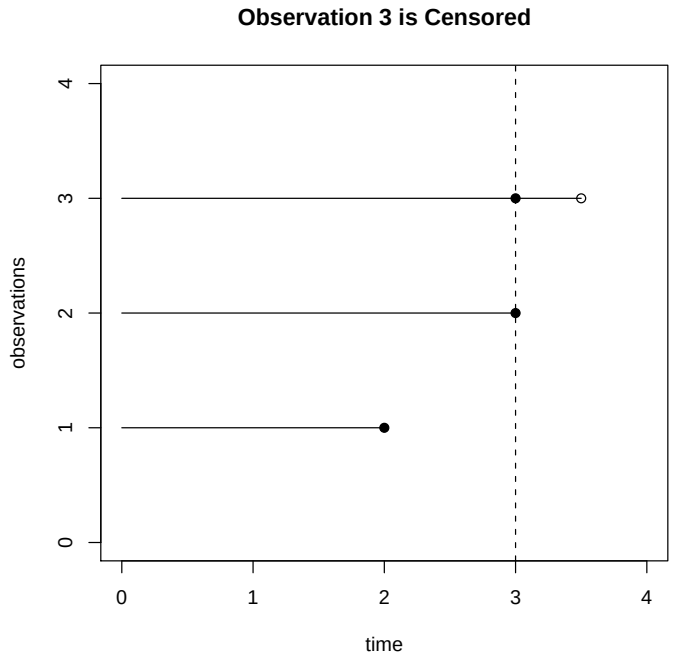
- Hiking trip!
  - Saturday April 25 at 9am at (probably) Breakheart Reservation
  - Meet in front of CGIS
  - More info on the Canvas message board, including an RSVP which will help coordinate rides

Outline

Review from last week

Three reasons we use duration models:

1. OLS assumes Y is Normal but duration dependent variables are always positive (number of years, number of days. etc.)
2. Duration models can handle time-varying covariates
3. Duration models can handle censoring



Duration Model Notation

T: a continuous, positive random variable representing the duration/survival times ($T = Y$)

f(t): the probability density function of T (the stochastic component)

F(t): the CDF of $f(t)$, $\int_0^t f(u)du = P(T \leq t)$, which is the probability of an event occurring before (or at exactly) time t

Survivor function: the probability of surviving (i.e. no event occurring) until at least time t: $S(t) = 1 - F(t) = P(T > t)$

Duration Model Notation

Hazard function or hazard rate: the probability of an event at time t given survival up to time t:

$$
\begin{aligned}
h(t) &= P(t \leq T < t + \tau | T \geq t) \\
&= P(\text{event at } t | \text{survival up to } t) \\
&= \frac{P(\text{survival up to } t | \text{event at } t) P(\text{event at } t)}{P(\text{survival up to } t)} \\
&= \frac{P(\text{event at } t)}{P(\text{survival up to } t)} \\
h(t) &= \frac{f(t)}{S(t)}
\end{aligned}
$$

Duration Model Notation

Therefore:

$$\underbrace{f(t)}_{\text{density function}} = \underbrace{h(t)}_{\text{hazard function}} \cdot \underbrace{S(t)}_{\text{survival function}}$$

$$\underbrace{f(t)}_{\text{density function}} = \underbrace{\frac{f(t)}{S(t)}}_{\text{hazard function}} \cdot \underbrace{S(t)}_{\text{survival function}}$$

Handling censoring

We know that censored observations they survived at least until some observed time, t^c, and that their true duration, t is greater than or equal to t^c.

For each observation, let's create a censoring indicator variable, c_i, such that

$$c_i = \begin{cases} 1 & \text{if censored} \\ 0 & \text{if not censored} \end{cases}$$

Censoring

We can incorporate the information from the censored observations into the likelihood function.

$$\begin{aligned}
\mathcal{L} &= \prod_{i=1}^{n} [f(t_i)]^{1-c_i} \; [P(T_i \geq t_i^c)]^{c_i} \\
&= \prod_{i=1}^{n} [f(t_i)]^{1-c_i} \; [1 - F(t_i)]^{c_i} \\
&= \prod_{i=1}^{n} [f(t_i)]^{1-c_i} \; [S(t_i)]^{c_i}
\end{aligned}$$

Uncensored observations contribute to the density function and censored observations contribute to the survivor function in the likelihood.

Outline

Relationship between count and duration models

Count models and duration models are two sides of the same coin

If you have an event that you're interested in, say Prussian soldiers being kicked by horses

Then you could model the number of kicked soldiers in some period of time (count model)
Or you could model the amount of time between kicks (duration model)

Counts and Durations

- Events whose counts are distributed Poisson are said to have a Poisson arrival process
- The Poisson distribution has an interesting relationship with the exponential distribution
- Specifically, the amount of time between events that have an Poisson arrival process has an exponential distribution

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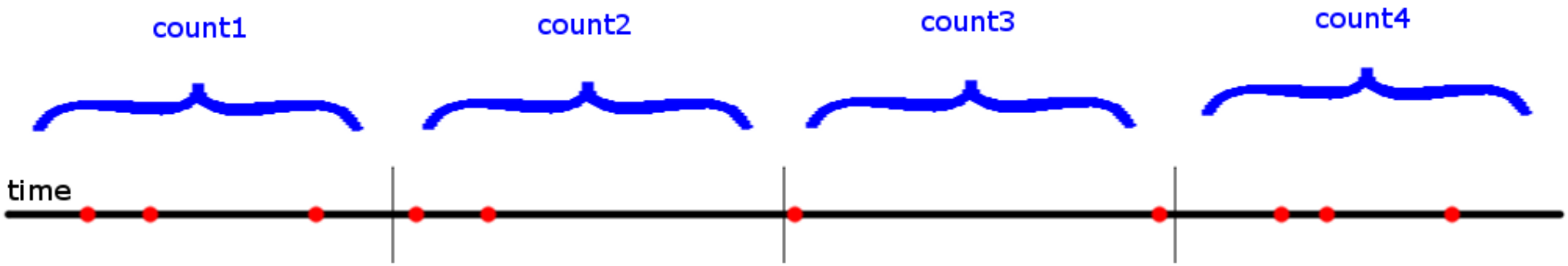
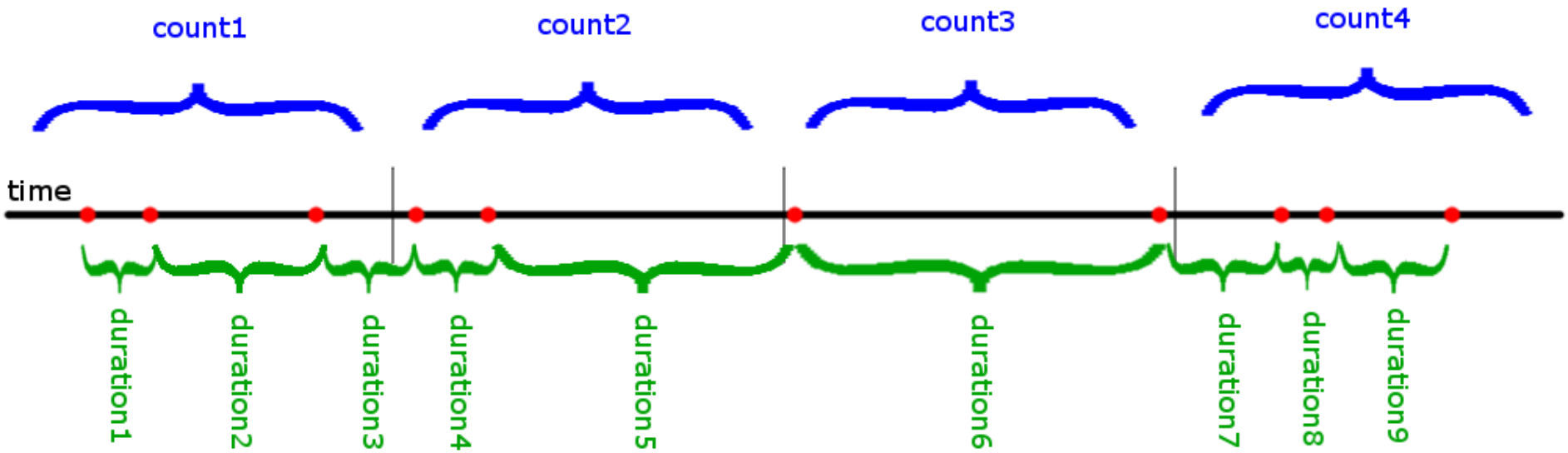


Counts and Durations

- Events whose counts are distributed Poisson are said to have a Poisson arrival process
- The Poisson distribution has an interesting relationship with the exponential distribution
- Specifically, the amount of time between events that have an Poisson arrival process has an exponential distribution



Principles of Poisson Process

- **Independent increments**: number of events occurring in two disjoint intervals is independent
- **Stationary increments**: the rate of occurrence is constant, so the distribution of number of occurrences depends only on the time length of interval
- Events occur at rate λ (expected occurrences per unit of time)
- $N_\tau =$ number of arrivals in time period of length τ
- $N_\tau \sim \text{Poisson}(\lambda\tau)$
- **Memorylessness property**: how much you have waited already is irrelevant

$$P(T > t + k | T > t) = P(T > k)$$

$$P(T > 8 | T > 3) = P(T > 5)$$

Two Possible Parameterizations of the Exponential Model

- $\lambda_i > 0$ is the **rate** parameter

$$T_i \sim \text{Exponential}(\lambda_i)$$

$$f(t_i) = \lambda_i e^{-\lambda_i t_i}$$

$$E(T_i) = \frac{1}{\lambda_i}$$

- $\theta_i > 0$ is **scale** parameter ($\theta_i = \frac{1}{\lambda_i}$)

$$T_i \sim \text{Exponential}(\theta_i)$$

$$f(t_i) = \frac{1}{\theta_i} e^{-\frac{t_i}{\theta_i}}$$

$$E(T_i) = \theta_i$$

Link Functions

- If you use a rate parameterization with λ_i:

$$E(T_i) = \frac{1}{\lambda_i} = \frac{1}{\exp(x_i\beta)}$$

  Positive β implies that expected duration time decreases as x increases.

- If you use a scale parameterization with θ_i

$$E(T_i) = \theta_i = \exp(x_i\beta)$$

  Positive β implies that expected duration time increases as x increases.

Hazard Function for Rate Parametrization

For $T_i \sim \text{Exponential}(\lambda_i)$:

$$
\begin{aligned}
f(t) &= \lambda_i e^{-\lambda_i t} \\
S(t) &= 1 - F(t) \\
&= 1 - (1 - e^{-\lambda t}) \\
&= e^{-\lambda_i t} \\
h(t) &= \frac{f(t)}{S(t)} \\
&= \frac{\lambda_i e^{-\lambda_i t}}{e^{-\lambda_i t}} \\
&= \lambda_i
\end{aligned}
$$

Hazard Function for Scale Parametrization

For $T_i \sim \text{Exponential}(\theta_i)$:

$$
\begin{aligned}
f(t) &= \frac{1}{\theta_i} \exp[-\frac{t}{\theta_i}] \\
S(t) &= 1 - F(t) \\
&= 1 - (1 - \exp[-\frac{t}{\theta_i}]) \\
&= \exp[-\frac{t}{\theta_i}] \\
h(t) &= \frac{f(t)}{S(t)} \\
&= \frac{\frac{1}{\theta_i} \exp[-\frac{t}{\theta_i}]}{\exp[-\frac{t}{\theta_i}]} = \frac{1}{\theta_i}
\end{aligned}
$$

Let's work with the scale parametrization

- Note that the hazard does not depend on t!
  - The exponential model thus assume a **flat hazard**: Every observation has its own hazard rate, but it does not change over time
  - Connected to **memorylessness property** of the exponential distribution

$$h(t) = \frac{1}{\theta_i} = \exp[-x_i\beta]$$

Positive β implies that hazard decreases and average survival time increases as x increases.

Estimation via ML:

$$
\begin{aligned}
\mathcal{L} &= \prod_{i=1}^{n} [f(t_i)]^{1-c_i} [1 - F(t_i)]^{c_i} \\
&= \prod_{i=1}^{n} \left[\frac{1}{\theta_i} e^{-\frac{t_i}{\theta_i}} \right]^{1-c_i} \left[e^{-\frac{t_i}{\theta_i}} \right]^{c_i} \\
\ell &= \sum_{i=1}^{n} (1 - c_i)(\ln \frac{1}{\theta_i} - \frac{t_i}{\theta_i}) + c_i(-\frac{t_i}{\theta_i}) \\
&= \sum_{i=1}^{n} (1 - c_i)(\ln e^{-\mathbf{x}_i\beta} - e^{-\mathbf{x}_i\beta} t_i) + c_i(-e^{-\mathbf{x}_i\beta} t_i) \\
&= \sum_{i=1}^{n} (1 - c_i)(-\mathbf{x}_i\beta - e^{-\mathbf{x}_i\beta} t_i) - c_i(e^{-\mathbf{x}_i\beta} t_i) \\
&= \sum_{i=1}^{n} (1 - c_i)(-\mathbf{x}_i\beta) - (1 - c_i)(e^{-\mathbf{x}_i\beta} t_i) - c_i(e^{-\mathbf{x}_i\beta} t_i) \\
&= \sum_{i=1}^{n} (1 - c_i)(-\mathbf{x}_i\beta) - e^{-\mathbf{x}_i\beta} t_i
\end{aligned}
$$

Quantities of interest

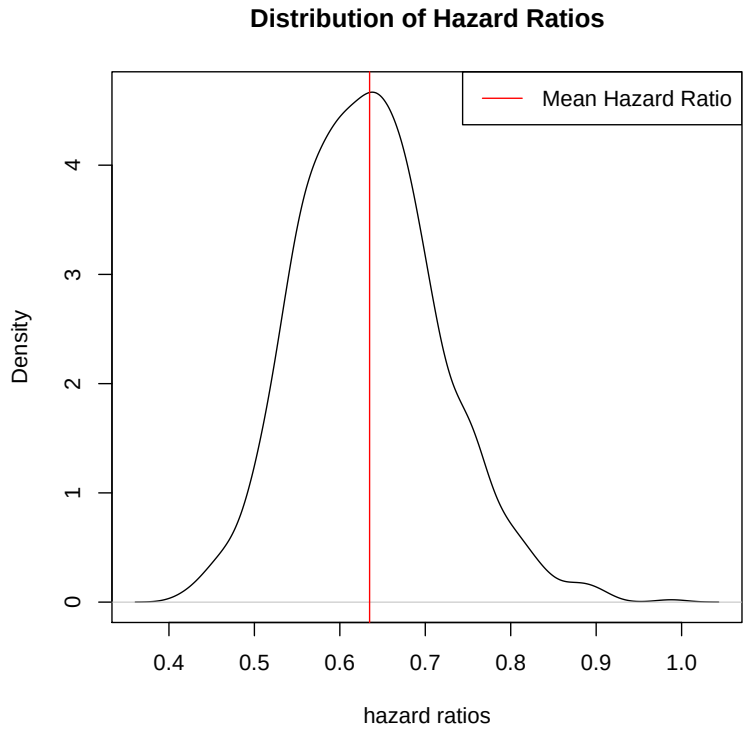
If our outcome variable is how long a parliamentary government lasts, we might be interested in the effect of majority versus minority governments. We could calculate:

- Hazard ratio of majority to minority governments
- Expected survival time for majority and minority governments
- Predicted survival times for majority and minority governments
- First differences in expected survival times between majority and minority governments

Hazard Ratios

$$
\begin{aligned}
\text{HR} &= \frac{h(t|\mathbf{x}_{\text{maj}})}{h(t|\mathbf{x}_{\text{min}})} \\
&= \frac{e^{-\mathbf{x}_{\text{maj}}\beta}}{e^{-\mathbf{x}_{\text{min}}\beta}} \\
&= \frac{e^{-\beta_0}e^{-x_1\beta_1}e^{-x_2\beta_2}e^{-x_3\beta_3}e^{-x_{\text{maj}}\beta_4}e^{-x_5\beta_5}}{e^{-\beta_0}e^{-x_1\beta_1}e^{-x_2\beta_2}e^{-x_3\beta_3}e^{-x_{\text{min}}\beta_4}e^{-x_5\beta_5}} \\
&= \frac{e^{-x_{\text{maj}}\beta_4}}{e^{-x_{\text{min}}\beta_4}} \\
&= e^{-\beta_4}
\end{aligned}
$$

Hazard ratio greater than 1 implies that majority governments fall faster (shorter survival time) than minority governments.

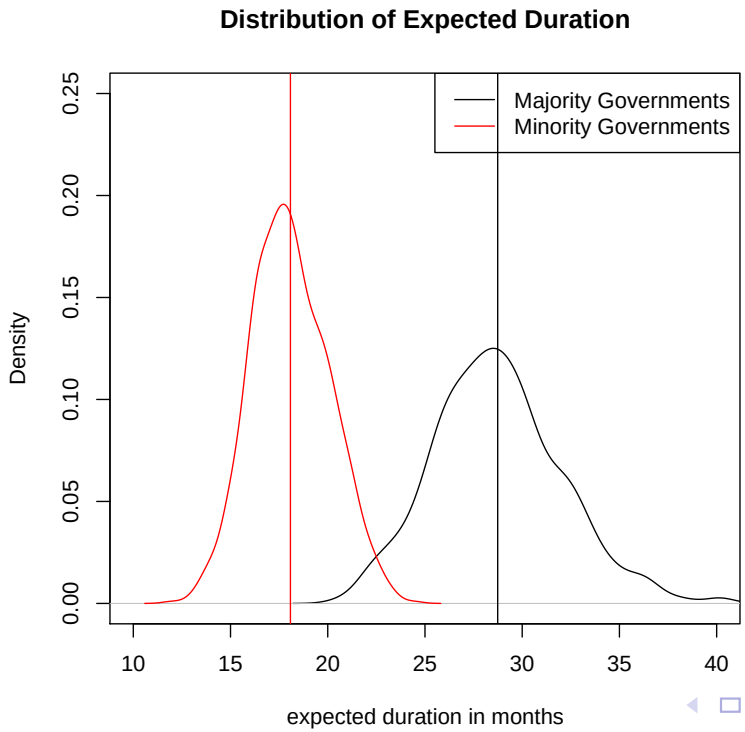


Majority governments survive longer than minority governments.

Expected (average) Survival Time

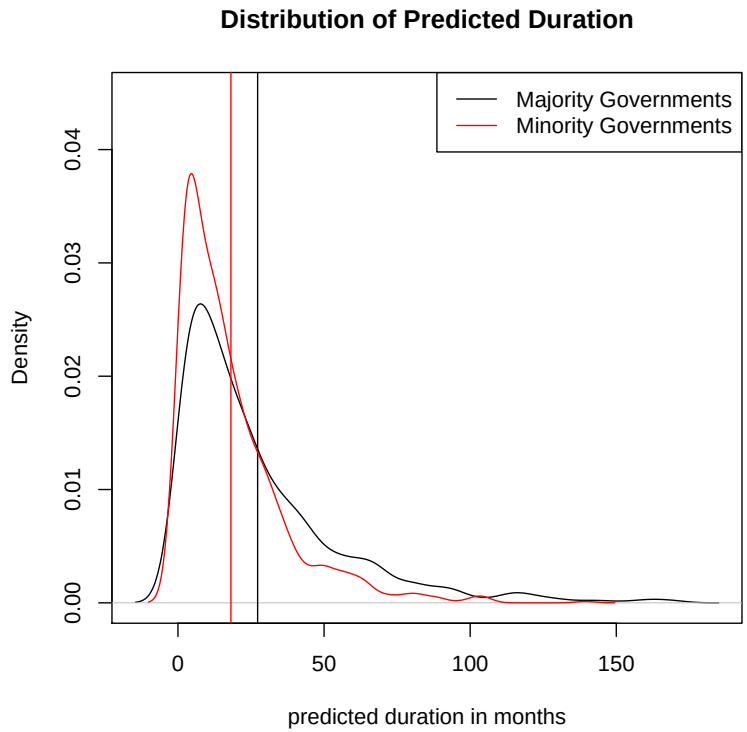
What's the expected survival time of a majority government compared to a minority one?
i.e. what is the sampling distribution of the mean survival for each government type?

$$E(T|\mathbf{x}_i) = \theta_i = \exp[\mathbf{x}_i\beta]$$



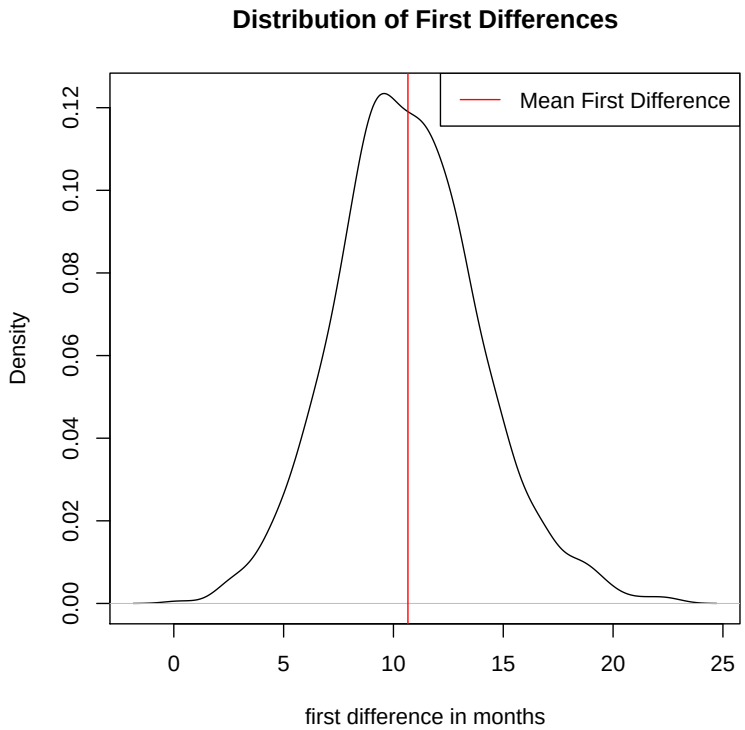
Predicted Survival Time

What is the predicted survival time of majority versus minority governments?
i.e. what is the distribution of the predicted duration? Draw predicted values from the exponential distribution.



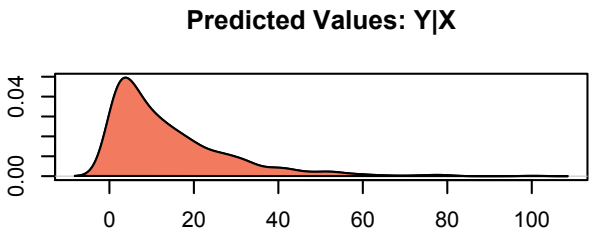
First Differences

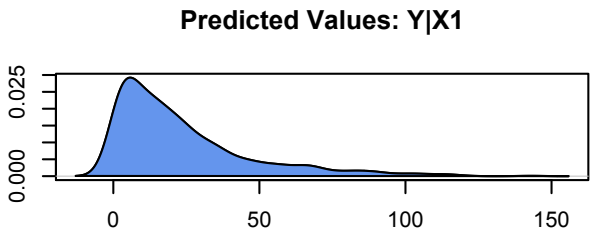
$$E(T|\mathbf{x}_{\text{maj}}) - E(T|\mathbf{x}_{\text{min}})$$

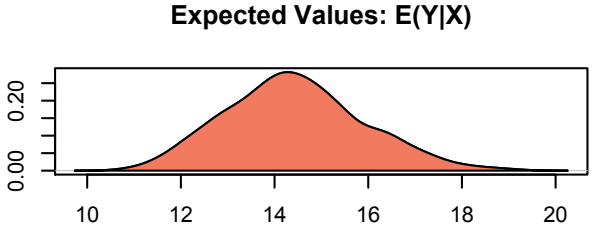


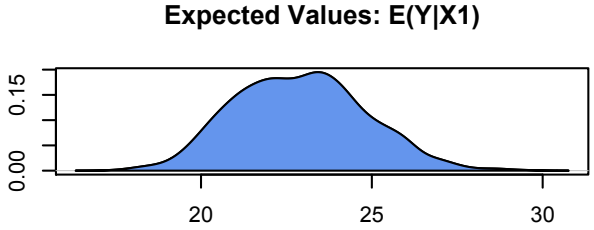
Quantities of Interest in Zelig

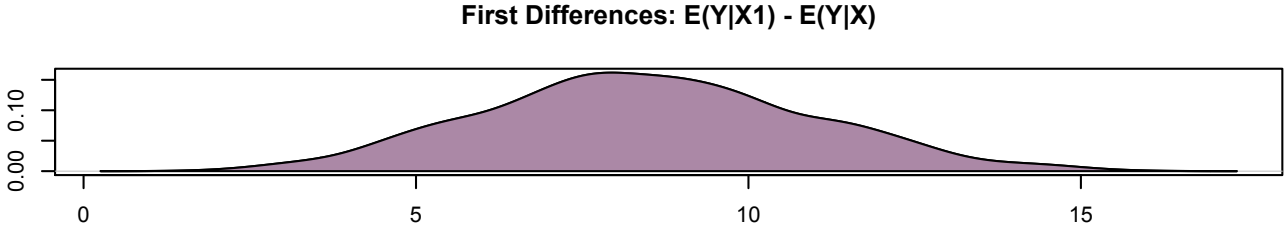
```
x.min <- setx(z.out,numst2=0)
x.maj <- setx(z.out,numst2=1)
s.out <- sim(z.out, x=x.min,x1=x.maj)
summary(s.out)
plot(s.out)
```

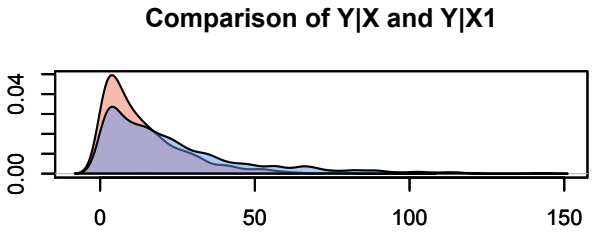

Predicted Values: Y|X
0.00
0.04
0
20
40
60
80
100

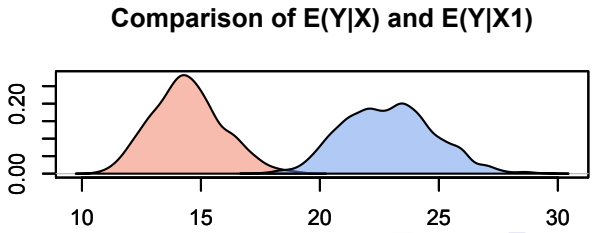

Predicted Values: Y|X1
0.000
0.025
0
50
100
150


Expected Values: E(Y|X)
0.00
0.20
10
12
14
16
18
20


Expected Values: E(Y|X1)
0.00
0.15
20
25
30


First Differences: E(Y|X1) - E(Y|X)
0.00
0.10
0
5
10
15


Comparison of Y|X and Y|X1
0.00
0.04
0
50
100
150


Comparison of E(Y|X) and E(Y|X1)
0.00
0.20
10
15
20
25
30

The exponential model is nice and simple, but the assumption of a flat hazard may be too restrictive.

What if we want to loosen that restriction by assuming a monotonic hazard?

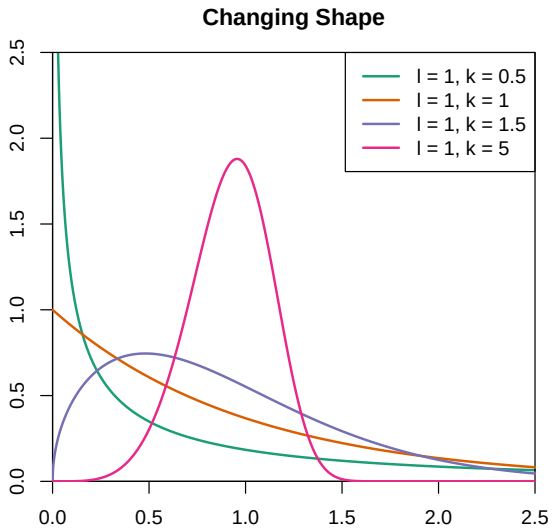
We can use the Weibull model.

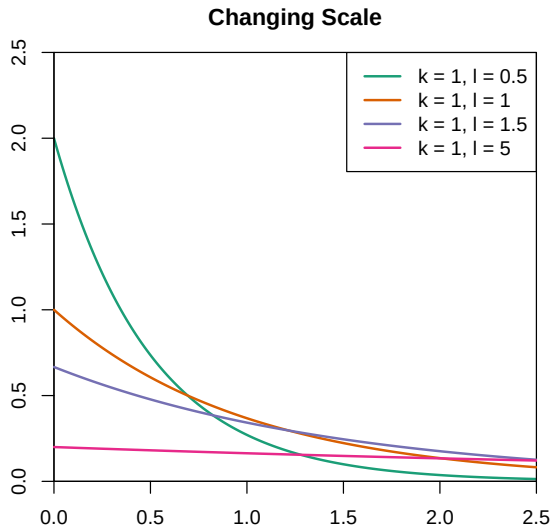
The Weibull Model

Similar to how we generalized the Poisson into a Negative Binomial by adding a parameter, we can do the same with the Exponential by turning it into a Weibull:

$$\begin{aligned} T_i &\sim \text{Weibull}(\lambda_i, \alpha) \\ E(T_i) &= \lambda_i \Gamma\left(1 + \frac{1}{\alpha}\right) \end{aligned}$$

$\alpha > 0$ is the shape parameter and $\lambda_i > 0$ is the scale parameter





The Weibull Model

$$f(t_i) = \left(\frac{\alpha}{\lambda_i^{\alpha}}\right) t_i^{\alpha-1} \exp\left[-\left(\frac{t_i}{\lambda_i}\right)^{\alpha}\right]$$

Model λ_i with covariates in the systematic component:

$$\lambda_i = \exp(x_i\beta)$$

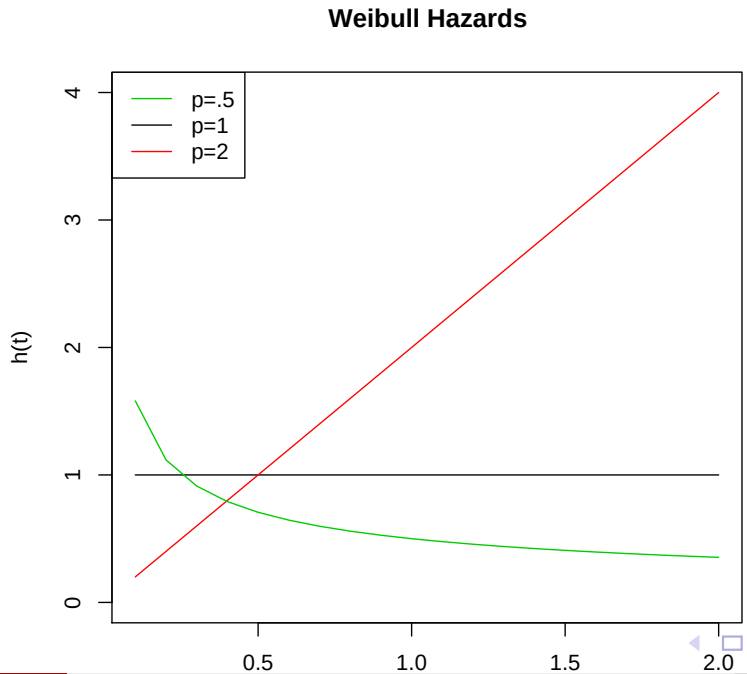
Positive β implies that expected duration time increases as x increases.

$$
\begin{aligned}
f(t_i) &= \left(\frac{\alpha}{\lambda_i^{\alpha}}\right) t_i^{\alpha-1} \exp\left[-\left(\frac{t_i}{\lambda_i}\right)^{\alpha}\right] \\
S(t_i) &= 1 - F(t_i) \\
&= 1 - (1 - e^{-(t_i/\lambda_i)^{\alpha}}) \\
&= e^{-(t_i)/\lambda_i)^{\alpha}} \\
h(t_i) &= \frac{f(t_i)}{S(t_i)} \\
&= \frac{\left(\frac{\alpha}{\lambda_i^{\alpha}}\right) t_i^{\alpha-1} \exp\left[-\left(\frac{t_i}{\lambda_i}\right)^{\alpha}\right]}{e^{-(t_i/\lambda_i)^{\alpha}}} \\
&= \left(\frac{\alpha}{\lambda_i}\right)\left(\frac{t_i}{\lambda_i}\right)^{\alpha-1} \\
&= \left(\frac{\alpha}{\lambda_i^{\alpha}}\right) t_i^{\alpha-1}
\end{aligned}
$$

Hazard monotonicity assumption

$h(t_i)$ is a function of λ_i, α and (in contrast to the exponential) t_i. Thus, the Weibull model relaxes the constant hazard assumption of the exponential and assumes a **monotonic hazard**.

- If $\alpha = 1$, $h(t_i)$ is flat and the model is the exponential model.
- If $\alpha > 1$, $h(t_i)$ is monotonically increasing.
- If $\alpha < 1$, $h(t_i)$ is monotonically decreasing.



The Cox Proportional Hazards Model

Pros:

- Makes no restrictive assumption about the shape of the hazard.
- A better choice if you want to know effects of the covariates and the nature of time dependence isn't important.

Cons:

- Only quantities of interest are hazard ratios.
- Can be subject to overfitting
- Shape of hazard is unknown (although there are semi-parametric ways to derive the hazard and survivor functions)

Other Parametric Models

- Gompertz model: monotonic hazard
- Log-logistic or log-normal model: nonmonotonic hazard
- Generalized gamma model: nests the exponential, Weibull, log-normal, and gamma models with an extra parameter

More resources about survival modeling

Box-Steffensmeier, Janet M. and Bradford S. Jones. 2004. *Event History Modeling.* Cambridge University Press.

King, Gary, James E. Alt, Nancy E. Burns, and Michael Laver. 1990. "A Unified Model of Cabinet Dissolution in Parliamentary Democracies." *American Journal of Political Science* 34(3): 846-971

Long, S. J. (1997) *Regression Models for Categorical and Limited Dependent Variables*. Thousand Oaks, CA: SAGE Publications, Inc.

McCullagh, Peter; Nelder, John (1989). *Generalized Linear Models*, Second Edition. Boca Raton: Chapman and Hall/CRC

Lam, Patrick. Survival Model Notes. http://www.patricklam.org/teaching.html

Outline

Setup

- Let's denote treatment as $T \in 0, 1$. $T = 1$ is treated group, $T = 0$ is control group.
- We have an outcome Y
- SUTVA: stable unit treatment value assumption
  - No interference between units - i.e. units don't talk to each other about the experiment
  - No hidden levels of treatment
- We have 2 potential outcomes per unit:

$$Y_i(1) \text{ and } Y_i(0)$$

- Before we think about how treatment was assigned, we can express this information in a potential outcomes table

Potential Outcomes Table

i	Name	$Y_i(1)$	$Y_i(0)$
1	Angela Merkel	$Y_1(1)$	$Y_1(0)$
2	Toucan Sam	$Y_2(1)$	$Y_2(0)$
3	Anne Boleyn	$Y_3(1)$	$Y_3(0)$
4	Hamburger Helper Mitt	$Y_4(1)$	$Y_4(0)$
5	Dr. Phil	$Y_5(1)$	$Y_5(0)$
6	Herschel Walker	$Y_6(1)$	$Y_6(0)$

Individual causal effect for Herschel Walker: $Y_1(1)$-$Y_1(0)$

All potential outcomes are fixed pre-treatment characteristics of individuals

Fundamental Problem of Causal Inference

i	Name	$Y_i(1)$	$Y_i(0)$	T_i
1	Angela Merkel	$Y_1(1)$	?	1
2	Toucan Sam	$Y_2(1)$	?	1
3	Anne Boleyn	?	$Y_3(0)$	0
4	Hamburger Helper Mitt	$Y_4(1)$	?	1
5	Dr. Phil	?	$Y_5(0)$	0
6	Herschel Walker	?	$Y_6(0)$	0

The **fundamental problem of causal inference** is that we only observe one potential outcome per unit!

Fundamental Problem of Causal Inference

How do we estimate the average treatment effect (ATE) from observed data?

$$\mathrm{E}[Y_i(1) - Y_i(0)]$$

Causal inference is a missing data problem! So we'll have to impute missing potential outcomes.

Unconfoundedness

To do this, we generally make another assumption: **unconfoundedness** of treatment,

$$P(T|Y(0), Y(1), X) = P(T)$$

Unconfoundedness translates to the following:

$$\mathrm{E}[Y(1)] = \mathrm{E}[Y(1)|T = 1] = \mathrm{E}[Y(1)|T = 0]$$
$$\mathrm{E}[Y(0)] = \mathrm{E}[Y(0)|T = 1] = \mathrm{E}[Y(0)|T = 0]$$

Therefore:

$$\begin{aligned} \mathrm{E}[Y(1) - Y(0)] &= \mathrm{E}[Y(1)] - \mathrm{E}[Y(0)] \\ &= E[Y(1)|T = 1] - E[Y(0)|T = 0] \\ &= E[Y^{obs}|T = 1] - E[Y^{obs}|T = 0] \end{aligned}$$

Causal Effects with Unconfoundedness

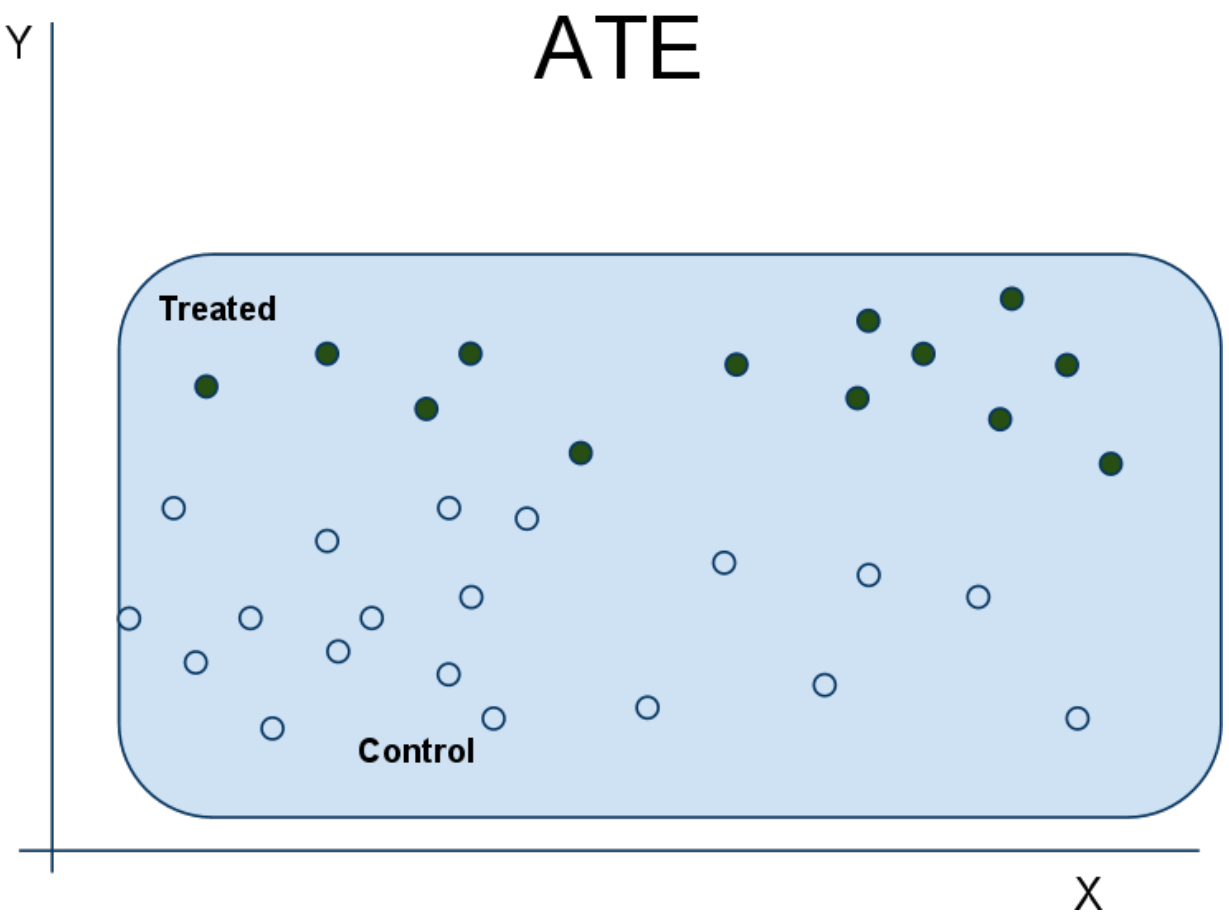
	T=1 (Treatment)	T=0 (Control)
$E[Y\|T=t]$	6.6	2.4

If this is our data, how would we estimate the average causal effect of T on Y? Assuming unconfoundedness:

$$\begin{aligned} \mathrm{E}[Y(1)-Y(0)] &= E[Y(1)|T=1]-E[Y(0)|T=0] \\ &= E[Y^{obs}|T=1]-E[Y^{obs}|T=0] \\ &= 6.6-2.4 \\ &= 4.2 \end{aligned}$$

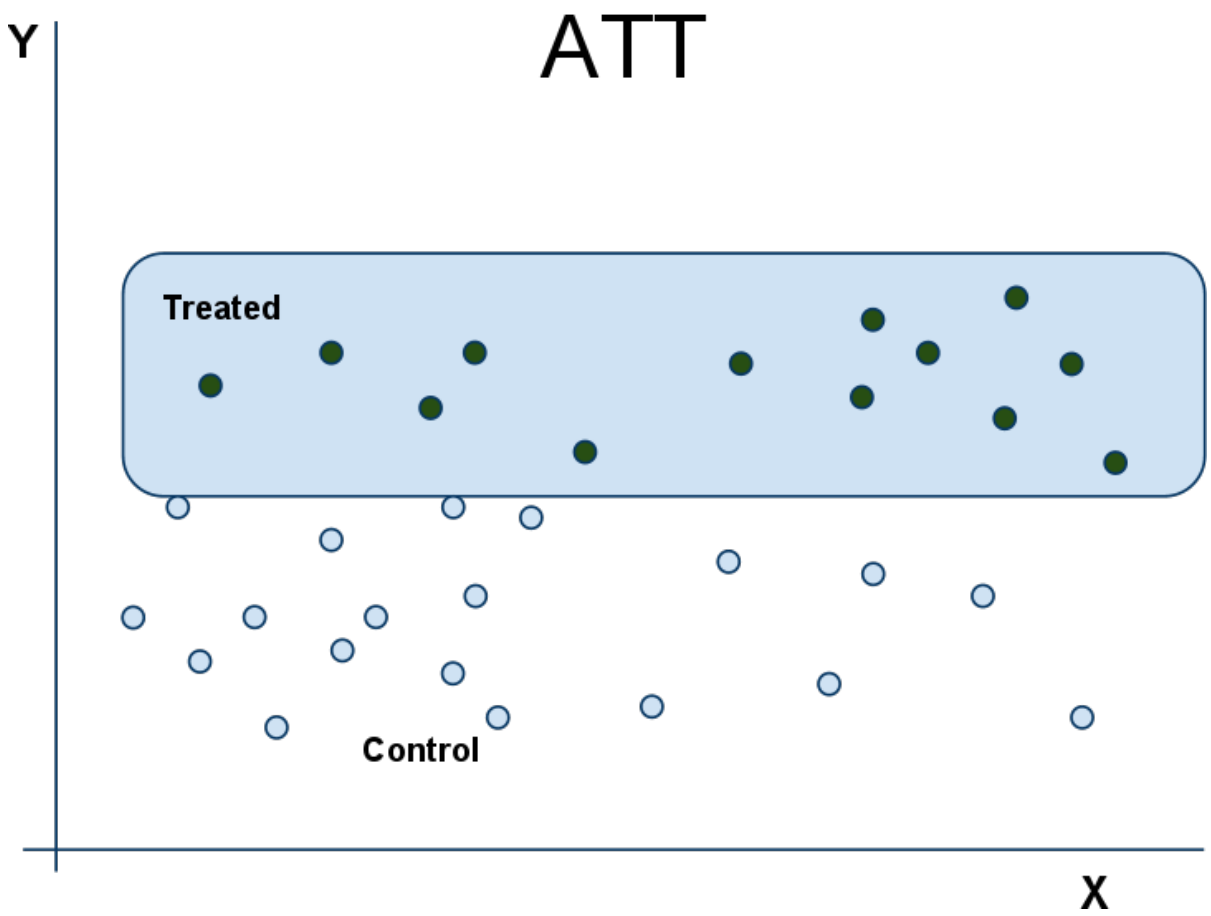
Possible estimators

Average Treatment Effect: need counterfactuals for all units



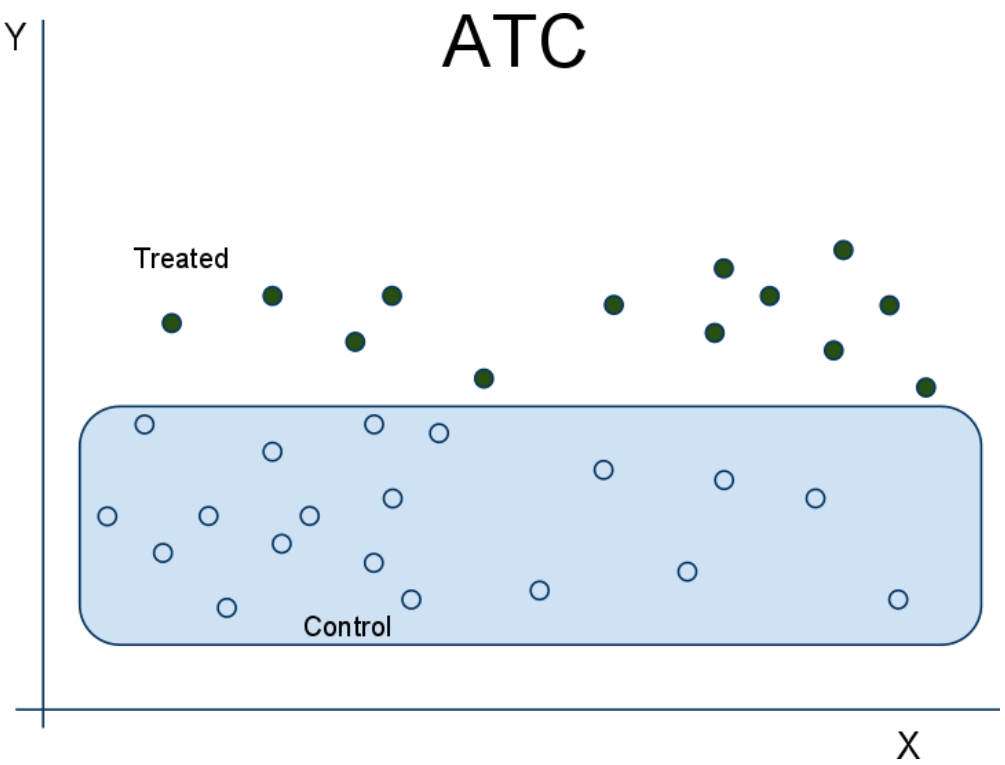
Possible estimators

Average Treatment Effect on the Treated:
need counterfactuals for all treated units



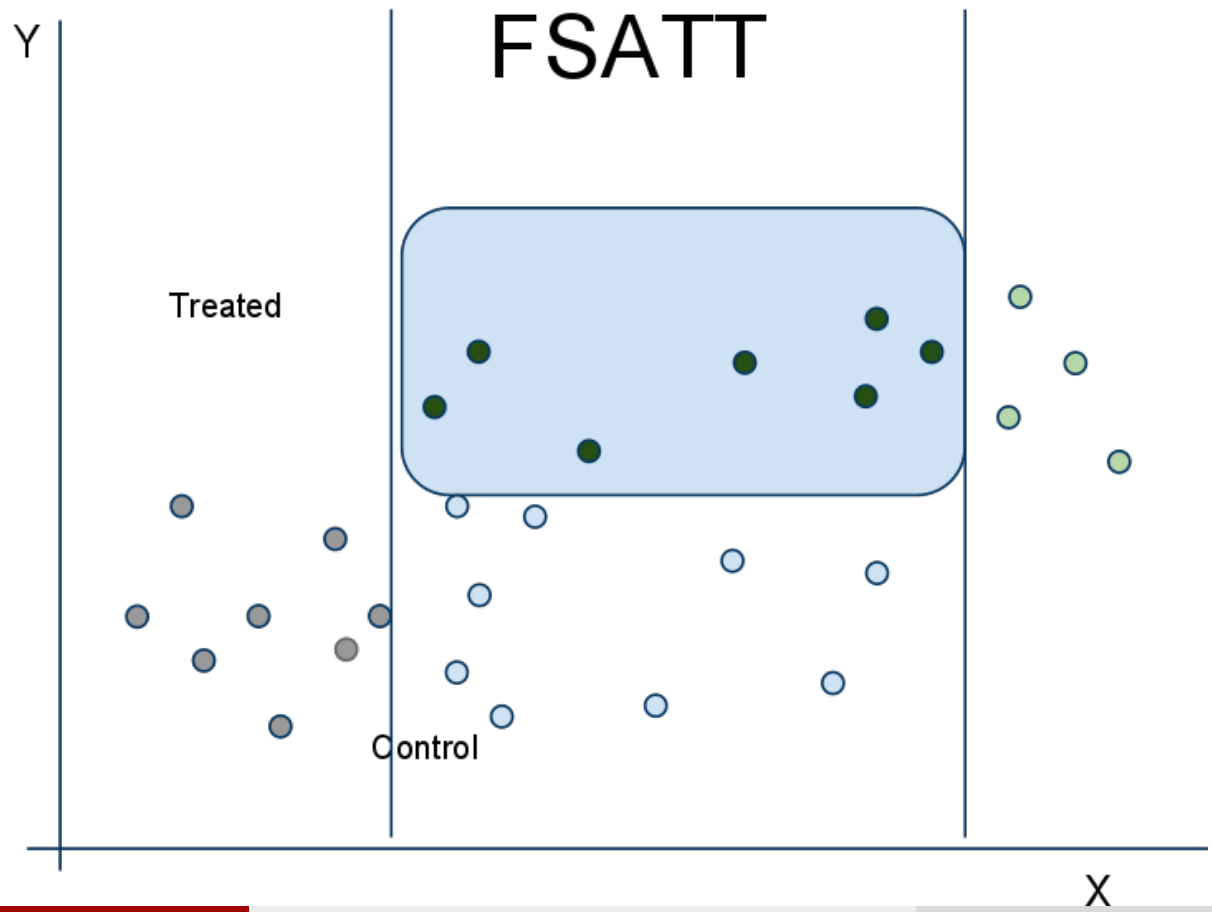
Possible estimators

Average Treatment Effect on the Controls:
need counterfactuals for all control units



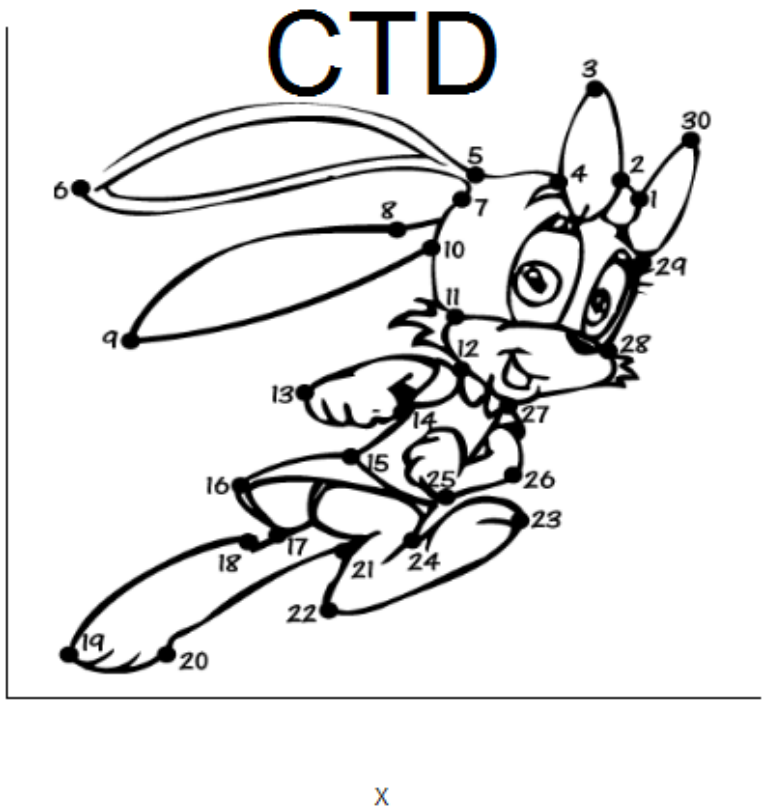
Possible estimators

Feasible Sample Average Treatment Effect on the Treated: need counterfactuals for all units which we can feasibly match to other units



Possible estimators

CTD:
Connect the dots



Yes, I did an online connect the dots puzzle while prepping these slides

Yes, I did screw up as I was doing the puzzle which apparently upset this cartoon bear:

Confounding (with *measured* covariates)

What if we have confoundedness?

$$P(T|Y(0), Y(1), X) \neq P(T)$$

$$P(T|Y(0), Y(1), X) = P(T|X)$$

- **Consequence**: imbalance between treated and control units in X
- **Possible solution**: matching

An example with confounding

Suppose that those who take the treatment are systematically different than those who don't take the treatment.

	T=1 (Treatment)	T=0 (Control)
$X = 0$	15	40
$X = 1$	35	10

In this case, $X = 0$ strata responds negatively to treatment and $X = 1$ strata responds positively to treatment. Moreover, it could be that those who are likely to be negatively affected by treatment are opting for control.

General Strategy of Matching

- 1. Condition on observed, pretreatment variables such that treatment assignment is uncorrelated with potential outcomes conditional on those covariates
- 2. Match data according to the strata defined by the values of these variables
- 3. Assess our matching procedure (check balance)

General Strategy of Matching

- 4a. Recombine these strata-specific causal effects into an overall treatment effect by weighting
- 4b. Proceed with parametric analysis (regression, t-test, etc.)
- 5. Sensitivity testing for either the confoundedness assumption or the parametric model.

Matching and Causal Inference

If you're interested in the causal effect of A on B, will the results of your parametric analysis be interpretable as a causal effect if you match?

NO $_t$

unless the covariates that you use to match perfectly characterize the data generation process

The purpose of matching for causal inference is to make your treatment seem as if it were truly randomly assigned

Get ready to be disappointed...

If you have unmeasured confounders: matching $\neq$ causal

If you have an omitted variable: matching $\neq$ causal

If you have the wrong functional relationship between confounders and treatment: matching $\neq$ causal

So when can matching help us to make causal statements?

Now the good news...

There's still instances where matching helps you make causal statements:

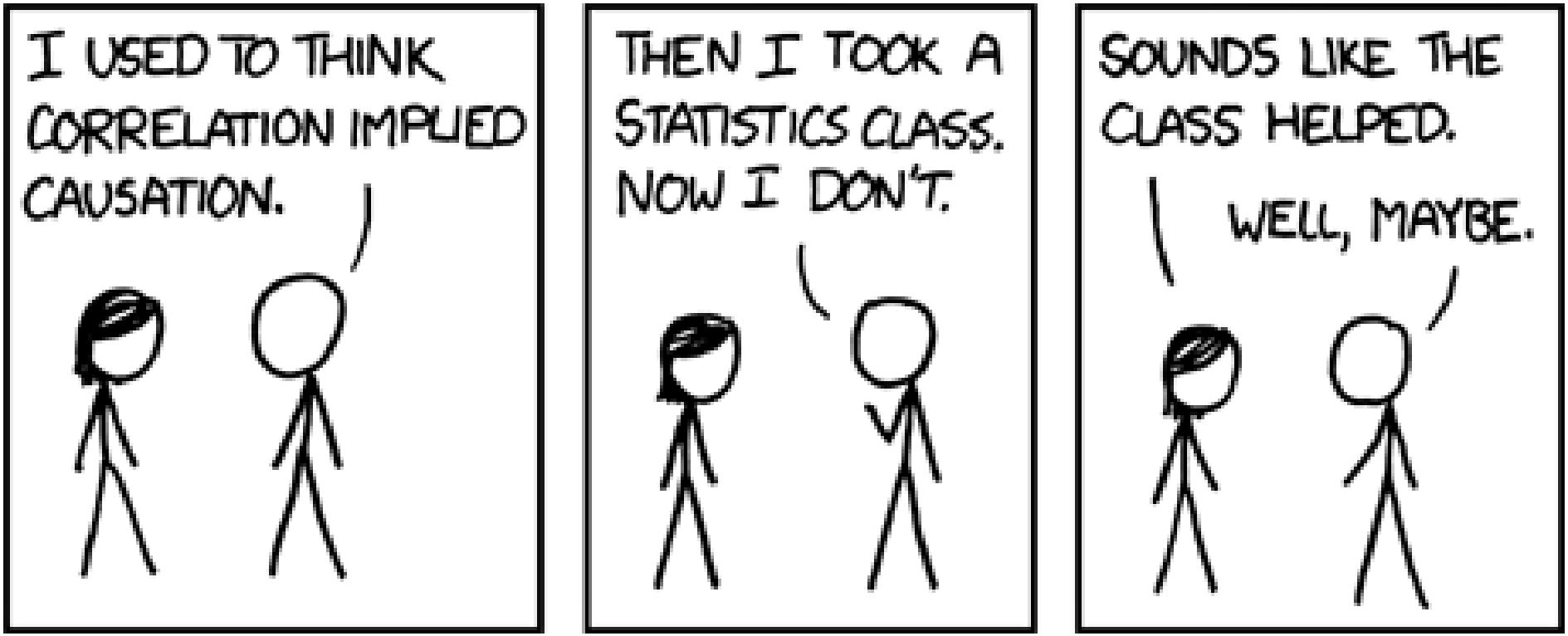
- You ran an experiment, and your randomization didn't work perfectly but you know where it went wrong
- You know all covariates which predict treatment, and you have them measured

Matching is still useful even is you're not making causal statements:

- Matching can help alleviate model dependence
- Matching can help deal with outliers
- Matching can help you understand the convex hull of your data and help you avoid extrapolating outside of it

Next week: Matching in R, balance checking

Questions?


I USED TO THINK CORRELATION IMPLIED CAUSATION.
THEN I TOOK A STATISTICS CLASS. NOW I DON'T.
SOUNDS LIKE THE CLASS HELPED.
WELL, MAYBE.